TELESCOPING SERIES REVISITED

QUIZ 1: $$\sum_{n=1}^{\infty} \frac{2}{4n^2-1} = \sum_{n=1}^{\infty} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right)$$

HANDWAVING

$$\boxed{\begin{aligned} S_n &= \left(\frac{1}{1} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{5}}\right) + \underbrace{\cdots}_{\text{ELLIPSIS}} + \left(\cancel{\frac{1}{2n-1}} - \frac{1}{2n+1}\right) \\ &= 1 - \frac{1}{2n+1} \end{aligned}}$$

$$S_n = \sum_{k=1}^{n} \left(\frac{1}{2k-1} - \frac{1}{2k+1} \right)$$

$$= \sum_{k=1}^{n} \frac{1}{2k-1} - \sum_{k=1}^{n} \frac{1}{2k+1}$$

$$\sum_{k=1}^{n} \frac{1}{2k-1} = \sum_{k=0}^{n-1} \frac{1}{2(k+1)-1}$$

INDEX SHIFT

UP 1 INSIDE

DOWN 1 LIMITS

$$= \sum_{k=0}^{n-1} \frac{1}{2k+1} - \sum_{k=1}^{n} \frac{1}{2k+1}$$

$$= \underbrace{\frac{1}{2(0)+1}}_{k=0} + \underbrace{\cancel{\sum_{k=1}^{n-1} \frac{1}{2k+1}}}_{k=1,\ldots,n-1} - \left(\underbrace{\cancel{\sum_{k=1}^{n-1} \frac{1}{2k+1}}}_{k=1,\ldots,n-1} + \underbrace{\frac{1}{2n+1}}_{k=n} \right)$$

$$= \frac{1}{1} - \frac{1}{2n+1}$$

$$= 1 - \frac{1}{2n+1}$$

11.10 TAYLOR SERIES

MATH 1A: LINEAR APPROXIMATION OF $f(x)$ NEAR $x=a$

$$f(x) \approx L(x) = f(a) + f'(a)(x-a) \to L'(x) = f'(a)$$

NOTE: $L(a) = f(a) + f'(a)(a-a) = f(a)$

$L'(a) = f'(a)$

POLYNOMIAL OF DEGREE 1 APPROXIMATE $f(x)$ NEAR $x=a$ SUCH THAT THE 0^{TH} + 1^{ST} DERIVATIVES OF THE APPROXIMATION FUNCTION MATCH THOSE DERIVATIVES OF ORIGINAL FUNCTION

WHY NOT INCREASE THE DEGREE OF APPROXIMATION TO GET MORE ACCURACY?

$$f(x) \approx T(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \text{ SUCH THAT } T^{(n)}(a) = f^{(n)}(a)$$

$$f^{(0)}(a) = T^{(0)}(a) = \sum_{n=0}^{\infty} c_n (a-a)^n = \sum_{n=0}^{\infty} c_n \cdot 0^n = c_0 0^0 = c_0 \cdot 1 = c_0 \to c_0 = f^{(0)}(a) = \frac{f^{(0)}(a)}{0!}$$

BY OUR DEF'N FOR TAYLOR/POWER SERIES

$$T^{(1)}(x) = \sum_{n=0}^{\infty} \frac{d}{dx} c_n (x-a)^n = \sum_{n=0}^{\infty} c_n n (x-a)^{n-1}(1)$$

$$= \sum_{n=0}^{\infty} n c_n (x-a)^{n-1} = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$$

$n=0 \to$ TERM $=0$

$$f^{(1)}(a) = T^{(1)}(a) = \sum_{n=1}^{\infty} n c_n (a-a)^{n-1} = \sum_{n=1}^{\infty} n c_n 0^{n-1} = 1 c_1 0^0 = 1 c_1 \longrightarrow c_1 = \frac{f^{(1)}(a)}{1} = \frac{f^{(1)}(a)}{1!}$$

$$T^{(2)}(x) = \frac{d}{dx} T^{(1)}(x) = \sum_{n=1}^{\infty} \frac{d}{dx} n c_n (x-a)^{n-1} = \sum_{n=1}^{\infty} n c_n (n-1)(x-a)^{n-2}(1)$$

$$= \sum_{n=\cancel{1}\,2}^{\infty} n(n-1) c_n (x-a)^{n-2}$$

WHEN $n=1$, TERM $=0$

$$f^{(2)}(a) = T^{(2)}(a) = \underline{2(1)} c_2 (1) = \underline{2}c_2 \longrightarrow c_2 = \frac{f^{(2)}(a)}{\underline{2}} = \frac{f^{(2)}(a)}{2!}$$

$$T^{(3)}(x) = \sum_{n=2}^{\infty} \frac{d}{dx} n(n-1) c_n (x-a)^{n-2}$$

$$= \sum_{n=3}^{\infty} n(n-1)(n-2) c_n (x-a)^{n-3}$$

$$f^{(3)}(a) = T^{(3)}(a) = \underline{3(2)(1)} c_3 (1) = \underline{6}c_3 \longrightarrow c_3 = \frac{f^{(3)}(a)}{\underline{6}} = \frac{f^{(3)}(a)}{3!}$$

NOTE: $0! = 1$

$$\left.\begin{aligned} 1! &= 1 \\ 2! &= 2 \cdot 1 \\ 3! &= 3 \cdot 2 \cdot 1 \end{aligned}\right\} \quad n! = n \cdot (n-1)!$$

$n=1$

$$\underline{1} = 1! = 1 \cdot 0! = \underline{0!}$$

$$0! = 1$$

CAN WE USE THE SAME PROPERTY TO FIND $(-1)!$

$$c_n = \frac{f^{(n)}(a)}{n!}$$

$$f(x) \approx \underbrace{T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n}_{\text{TAYLOR SERIES}}$$

IF $f(x) = T(x)$ WE SAY $f(x)$ IS ANALYTIC